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A Study of Connectivity for Hyperspaces in Cyclic Order

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ABSTRACT: An arc is a homeomorphic path, which is a continuous function in a space. In this paper, we have studied arcwise connectivity for the spaces which have closed connected subsets in a way that, these are separated by disjoined open sets.

Keywords: Hyperspace; homeomorphic path; arcwise connectivity; continuum and induction theory.

INTRODUCTION: We start by briefly introducing the context. Throughout this paper X denotes a non degenerate compact connected metric space. The class of all spaces of non-empty closed connected subsets of X with Hausdorffness in metric function is represented by $\lambda(X)$.

Preliminaries: An arc is a homeomorphic path, which is a continuous function in a space. Let Z denote the space and a continuous function $\alpha: [0,1] \rightarrow Z$ a path in it. It is apparent that the existence of a path α in a Hausdorff space implies the existence if an arc β such that $\beta(0) = \alpha(0), \beta(1) = \alpha(1)$ and the range of β is contained in the range of α. An order arc $\lambda(X)$ is an arc α in $\lambda(X)$ which is a monotone function wherefrom either $\alpha(t) \subset \alpha(t')$ whenever $t < t'$ or $\alpha(t) \supset \alpha(t')$ whenever $t \geq t'$. Our endeavor is to prove:

Theorem 1: If $A_1, A_2, \ldots, A_n$ are distinct members of $\lambda(X)$ and each of the sets $\lambda(X) - A_i$ is arcwise connected, then $\lambda(X) - C\{A_1, \ldots, A_n\}$ is arcwise connected.

The earlier proofs pose a question. If $\mathfrak{I}$ is a compact totally disconnected subset of $\lambda(X)$ and if $\lambda(X) - A$ is arcwise connected for each $A \in \mathfrak{I}$, does not follows that $\lambda(X) - \mathfrak{I}$ is arcwise connected? To answer these questions, we give the following lemmas:

Lemma 1.1: If A and B belong to a class $\lambda(X)$ and if A is contained in B, then there exist an order arc α in $\lambda(X)$ such that $\alpha(0) = A$ and $\alpha(1) = B$.

Proof - The proof is obvious.

Lemma 1.2: If α is a path in $\lambda(X)$ and β is defined on $[0,1]$ by $\beta(t) = \cup\{\alpha(s): 0 \leq s \leq t\}$,
then β is a path in $\lambda(X)$.

Proof - β takes its values in $\lambda(X)$ by (1.2) of [5]. The continuity of β is obvious.

Lemma 1.3: Suppose $Y_1, \ldots\ldots, Y_n$ and A are distinct members of $\lambda(X)$ and α is an arc in $\lambda(X) - \{Y_1, \ldots\ldots, Y_n\}$, such that $\alpha(0) - A \neq \emptyset \neq \alpha(1) - A$, then there is an arc β in $\lambda(X) - \{Y_1, \ldots\ldots, Y_n\}$ such that $\beta(0) = \alpha(0)$ and $\beta(1) = \alpha(1)$.

Proof - Without loss of generality, we may suppose that there exist $k \in \{0,1, \ldots, n\}$, such that $Y_i \subset A$, if $i \leq k$ and $Y_i - A \neq \phi$, if $i > k$.

The lemma is immediate, if $\alpha(t) \neq A, \forall\, t \in [0,1]$, so we suppose $\alpha(t_1) = A$, where $0 < t_1 < 1$. Now we define

$$t_0 = \inf\{t \leq t_1 : t \leq s \leq t_1, then\; \alpha(s) \subset A\}$$
$$t_2 = \sup\{t \geq t_1 : t_1 \leq s \leq t, then\; \alpha(s) \subset A\}.$$

Since $\alpha(0)$ and $\alpha(1)$ each contain points outside A, we see that $0 < t_0 \leq t_1 \leq t_2 < 1$. Further, there exists an open set U, such that $U \cap \gamma_i \neq \phi$ for each $i > k$ and $\bar{U} \cap A = \phi$ and there exist $\epsilon > 0$ and $\epsilon' > 0$ with $0 < t_0 - \epsilon, t_2 + \epsilon' < 1$ and satisfying

$$\alpha(t_0 - \epsilon) - A \neq \emptyset \neq \alpha(t_2 + \epsilon') - A.$$

If $t_0 - \epsilon \leq t < t_0$, then $\alpha(t) \cap \bar{U} = \phi$ and if $t_2 < t \leq t_2 + \epsilon'$, then $\alpha(t) \cap \bar{U} = \phi$. At this point, let $t_3 = \frac{t_2 + \epsilon' + 1}{2}$ and define

$$\bar{\alpha}(t) = \begin{cases} \alpha(t), & if\; 0 \leq t \leq t_2 + \epsilon', \\ \alpha(t_2 + \epsilon'), & if\; t_2 + \epsilon' \leq t \leq t_3, \\ \alpha(2t - 1), & if\; t_3 \leq t \leq 1. \end{cases}$$

Thus $\bar{\alpha}$ is a path which traces the same arc as α, but $\bar{\alpha}$ is situated on the point $\alpha(t_2 + \epsilon')$ over the interval $[t_2 + \epsilon', t_3]$.

We now define:

$$\beta(t) = \begin{cases} \alpha(t), & if\ 0 \le t \le t_0 - \epsilon', \\ \cup\{\alpha(s): t_0 - s \le s \le t\}, & if (t_0 - \epsilon) \le t \le (t_2 + \epsilon') \end{cases}$$

On the interval $[t_2 + \epsilon', t_3]$ where β is our desired path, which is a decreasing order arc from $\beta(t_2 + \epsilon')$ to $\alpha(t_2 + \epsilon')$ and $\beta(t) = \bar{\alpha}(t)$.

We now see that, lemma 1.1 suggests that $\beta[t_0 - \epsilon, t_2 + \epsilon']$ is a path in $\lambda(X)$ and hence β is a path in $\lambda(X)$. Clearly, $\beta(0) = \alpha(0)$ and $\beta(1) = \alpha(1)$. Since $\alpha(t_0 - \epsilon) - A$ and $\alpha(t_2 + \epsilon') - A$ are non-empty, it follows that $\beta(t) - A$ is non-empty, if $t_0 - \epsilon \le t \le t_3$. Consequently, all values of $\beta(t)$ are distinct from A and Y_i $(i \le k)$. Moreover, since $\alpha(t) \cap \bar{U} = \phi$ in $[t_0 - \epsilon, t_2 + \epsilon']$, we see that $\beta(t) \cap \bar{U} = \phi$ on the same interval and hence all the value of $\beta(t)$ are distinct from $Y_i (i > k)$. This completes the proof of the lemma.

We now claim that lemma 1.3 is true when $\{Y_1, \dots \dots, Y_n\}$ is replaced by an arbitrary closed subset of $\lambda(X)$.

Now in order to prove the main result in theorem 1, we consider first the case when all the sub continua A_i are proper.

Theorem 2: If $A_1, \dots, A_n$ are distinct members of $\lambda(X) - X$ and if each of the sets $\lambda(X) - \{A_i\}$ is arcwise connected, then $\lambda(X) - \{A_1, \dots, A_n\}$ is arcwise connected.

Proof - If $n = 1$ then the proof is trivial, so we assume $n > 1$. Now applying induction, we say that the theorem is true for $n - 1$ inembers of $\lambda(X) - \{X\}$. It suffices to show that, if $k \in \lambda(X) - \{A_1, \dots, A_n, X\}$, then there is a path β in $\lambda(X) - \{A_1, \dots, A_n\}$ with $\beta(0) = K$ and $\beta(1) = X$.

Case 1: There exists i such that $K - A_i \neq \phi$. We may assume $i = 1$. Again, by the application of induction theory, there is an arc α in $\lambda(X) - \{A_2, \dots, A_n\}$, such that $\alpha(0) = K$ and $\alpha(1) = X$. By lemma 3.3 existence of β is proved, which is the desired path.

Case 2: $K \subset \{A_1 \cap \dots \cap A_n\}$. We may assume $i = 1$. By the induction theorem, there is an arc α in $\lambda(X) - \{A_2, \dots, A_{n-1}\}$, such that $\alpha(0) = K$ and $\alpha(1) = A_n$. Since $\alpha(1) - A_i$ is non-empty, there exists $t_1 < 1$, such that $\alpha(t_1) - A_1 \neq \emptyset$; moreover, $\alpha[0, t_1]$ lies in $\lambda(X) - \{A_1, \dots, A_n\}$. By case 1, there is a continuation γ in $\lambda(X) - \{A_1, \dots, A_n\}$, such that $\gamma(0) = \alpha(t)$ and $\gamma(1) = X$. If we define

$$\beta(t) = \begin{cases} \alpha(2t, t), & if\ \ 0 \le t \le \frac{1}{2} \\ \gamma(2t - 1), & if\ \ \frac{1}{2} \le t \le 1, \end{cases}$$

Then it can be easily verified that β is a path from K to X in $\lambda(X) - \{A_1, \dots, A_n\}$.

Proof of Theorem 1: Restricting the conditions in the proof of the theorem 2, we assume further that $A_n = X$. Let K_0 and K_1 be distinct members of $\lambda(X) - \{A_1, \dots, A_{n-1}, X\}$.

By theorem 2, there is an arc β in $\lambda(X) - \{A_1, \dots, A_{n-1}\}$ with $\beta(0) = K_0$ and $\beta(1) = K_1$. Then either β is the desired path or there exists t_1 with $\beta(t_1) = X$; in the latter case, there are $t_0 < t_1$ and $t_2 > t_1$, such that

$$\beta(t_0) - A_1 \neq \phi \neq \beta(t_2) - A_1$$

and both $\beta[0, t_0]$ and $\beta[t_2, 1]$ lie in $\lambda(X) - \{A_1, \dots, A_{n-1}, X\}$.

By the induction theory, there is an arc α in $\lambda(X) - \{A_2, \dots, A_{n-1}, X\}$ with $\alpha(0) = \beta(t_0)$ and $\alpha(1) = \beta(t_2)$ and by lemma 1.3, we may assume α in $\lambda(X) - \{A_2, \dots, A_{n-1}, X\}$. If we define γ by

$$\gamma(t) = \begin{cases} \beta(3t_0 t), & if\ 0 \le t \le \frac{1}{3} \\ \alpha(3t - 1), & if \frac{1}{3} \le t \le 2/3 \\ \beta(3t_2(1 - t) + 3t - 2), & if\ \ 2/3 \le t \le 1. \end{cases}$$

Then it is easily verified that γ is a path in $\lambda(X) - \{A_1, \dots, A_{n-1}, X\}$ with $\gamma(0) = K_0$ and $\gamma(1) = K_1$.

CONCLUSION: On account of the usefulness, general character and significance in technical applications, it is hoped that, this research paper shall provide future scope for researchers and students in the field of connectivity in hyperspaces.

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