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Application of Product of Chebyshev Hermite Polynomials, *I*-Function and *M*- Series in Heat Conduction

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ABSTRACT: As an example of the application of *I*-function in applied mathematics, in this paper first we establish two integrals involving the product of *I*-function, *M*-series and Chebyshev Hermite polynomials and then we make its application to solve a boundary value problem on heat conduction.

Keywords: Chebyshev Hermite polynomials, heat conduction, integrals.

INTRODUCTION

The I-function defined by Saxena[8], has been further studied by other workers[6, 7 & 9]. In this paper, we will define and represent the I-function in the following manner:

$$I_{P_i,Q_i:R}^{M,N}[x] = I_{P_i,Q_i:R}^{M,N}\left[x \left| \begin{matrix} (a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,P_i} \\ (b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i} \end{matrix} \right.\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \phi(\xi)x^{\xi}d\xi\,, \qquad (1.1)$$

Where;

$$\phi(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j - \beta_j\xi)\ \prod_{j=1}^{N}\Gamma(1 - a_j + \alpha_j\xi)}{\sum_{i=1}^{R}\left\{\prod_{j=M+1}^{Q_i}\Gamma(1 - b_{ji} + \beta_{ji}\xi)\ \prod_{j=N+1}^{P_i}\Gamma(a_{ji} - \alpha_{ji}\xi)\right\}}. \qquad (1.2)$$

where $\mathcal{L}$ is the path of integration separating the increasing and decreasing sequences of the poles of the integrand and the convergence, existence conditions and other details of the I-function, given in the literature[9].

The M-Series is a particular case of the H-function of Inayat-Hussain[3]. The generalized hypergeometric function and Mittag-Laffer function follow as its particular case[4&5]. We defined the M-Series as follows:

$${}_pM_q^{\alpha}\ \left(a_1\dots.a_p;b_1\dots.b_q;x\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k\dots\dots\dots(a_p)_k}{(b_1)_k\dots\dots.(b_q)_k}\frac{x^k}{\Gamma(\alpha k+1)} \qquad (1.3)$$

Here $\alpha \epsilon$ C, R(α)> 0, and $(a_j)_k, (b_j)_k$ are pochammer symbols. The series (1.3) is defined when none of the denominator parameters b_j's, j = 1,2, ….. q is a negative integer or zero, if any parameters a_j is negative then the series (1.3) terminates into a polynomial in x. By using ratio test, it is evident that the series (1.3) is convergent for all x, when $\geq p$, it is convergent for $|x| < 1$ when $p = q + 1$, divergent when $p > q + 1$. In some case the series is convergent for $x = 1, x = -1$. Let us consider;

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j.$$

When $p = q + 1$,the series is absolutely convergent for $|x| = 1$ if R$(b) < 0$, convergent for $x = -1$, if $0 \leq R(b) < 1$, and divergent for $|x| = 1$, if $1 \leq R(b)$.

Formulae Required: In this paper, we shall make application of the following well known integrals:

$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2/2} He_{2n}(x)dx = \frac{2^{\rho+n+\frac{1}{2}}\Gamma\left(\rho+\frac{1}{2}\right)\Gamma(\rho+1)}{\Gamma(1-n+\rho)}, \qquad (2.1)$$

$$\int_{-\infty}^{\infty} x^{2\rho+1} e^{-x^2/2} He_{2n+1}(x)dx = \frac{2^{\rho+n+\frac{3}{2}}\Gamma\left(\rho+\frac{3}{2}\right)\Gamma(\rho+1)}{\Gamma(1-n+\rho)}. \qquad (2.2)$$

Where; $\rho \leq n, \rho = 0,1,2,\cdots; n = 0,1,2,\cdots$.

Integrals: The following integrals to be evaluated here:

$$\int_{-\infty}^{\infty} x^{2\rho} e^{-x^2/2} He_{2n}(x)\ {}_pM_q^{\tau}\left[(g_p);(h_q);ax^{2\mu}\right] I_{P_i,Q_i:R}^{M,N}[zx^{2\sigma}]\,dx = 2^{\rho+n+\frac{1}{2}}$$

$$\bullet \sum_{k=0}^{\infty} M(k) I_{P_i+2,Q_i+1:R}^{M,N+2}\left[z2^{\sigma}\left|\begin{matrix}\left(\frac{1}{2}-\rho-\mu k,\sigma\right),(-\rho-\mu k,\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(n-\rho-\mu k,\sigma)\end{matrix}\right.\right], \qquad (3.1)$$

Where; $\sigma > 0, \rho \leq n, \rho = 0,1,2,\cdots; n = 0,1,2,\cdots$ and

$$M(k) = \frac{\prod_{k=1}^{p}(g_j)_k}{\prod_{k=1}^{q}(h_j)_k}\frac{2^{\mu k}a^k}{\Gamma(\tau k+1)}. \qquad (3.2)$$

$$\int_{-\infty}^{\infty} x^{2\rho+1} e^{-x^2/2} He_{2n+1}(x)\ {}_pM_q^{\tau}\left[(g_p);(h_q);ax^{2\mu}\right] I_{P_i,Q_i:R}^{M,N}[zx^{2\sigma}]\,dx = 2^{\rho+n+\frac{3}{2}}$$

$$\bullet \sum_{k=0}^{\infty} M(k) I_{P_i+2,Q_i+1:R}^{M,N+2}\left[z2^{\sigma}\left|\begin{matrix}\left(-\frac{1}{2}-\rho-\mu k,\sigma\right),(-\rho-\mu k,\sigma),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(n-\rho-\mu k,\sigma)\end{matrix}\right.\right], \qquad (3.3)$$

Where; $\sigma > 0, \rho \leq n, \rho = 0,1,2,\cdots; n = 0,1,2,\cdots$ and $M(k)$ is given by (3.2).

To establish (3.1), we using series representation for M-series as given by equation (1.3), and the I-function in terms of Melline – Barnes type contour integral as given in (1.1), changing the order of integration and summation we find that

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\theta(\xi)\left[\sum_{k=0}^{\infty} M(k)\left\{\int_{-\infty}^{\infty} x^{2\rho+2\mu k+2\sigma\xi} e^{-x^2/2} He_{2n}(x)dx\right\}\right] z^{\xi}\,d\xi,$$

Evaluating the x - integral with the help of the integral (2.1) and finally interpreting it with (1.1), we arrive at the integral (3.1). The proof of the integral (3.3) can be developed in the similar manner.

Application to Heat Conduction: In this section, we consider a problem on heat conduction under certain conditions. We discuss the following equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2}\right) - \frac{1}{2}k\left(\frac{x^2}{2} - 1\right)u, \qquad (-\infty < x < \infty), \qquad (4.1)$$

Where; $u(x,t)$ tends to zero, if the value of t is maximum and when $|x| \to \infty$ and $u(x,0) \equiv u(x)$. Equation (4.1) related to the problem of heat conduction [2, p. 134 (4)]:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2}\right) - Hu, \qquad (4.2)$$

in infinite region $(-\infty < x < \infty)$, while $H = \frac{1}{2}k\left(\frac{x^2}{2} - 1\right)$.

Solution of the problem: We may assume that the general solution of partial differential equation (4.1) to be [1, p. 32 (5.1), p. 33 (5.3)]:

$$u_1(x,t) = \sum_{n=0}^{\infty} C_{2n} e^{-2knt - x^2/4}\, He_{2n}(x), \qquad (5.1)$$

Where;

$$C_{2n} = \frac{1}{(2n)!\,\sqrt{(2\pi)}} \int_{-\infty}^{\infty} u_1(x) e^{-x^2/4}\, He_{2n}(x)\, dx, \qquad (5.2)$$

and

$$u_2(x,t) = \sum_{n=0}^{\infty} C_{2n+1} e^{-k(2n+1)t - x^2/4}\, He_{2n+1}(x), \qquad (5.3)$$

Where;

$$C_{2n+1} = \frac{1}{(2n+1)!\,\sqrt{(2\pi)}} \int_{-\infty}^{\infty} u_2(x) e^{-x^2/4}\, He_{2n+1}(x)\, dx, \qquad (5.4)$$

Now we shall consider the problem of determining $u_1(x,t)$ and $u_2(x,t)$, where, let;

$$u_1(x,0) \equiv u_1(x) = x^{2\rho} e^{-x^2/4}\, {}_pM_q^{\tau}\left[(g_p);(h_q);ax^{2\mu}\right] I_{P_i,Q_i:R}^{M,N}\left[zx^{2\sigma}\right] \qquad (5.5)$$

and

$$u_2(x,0) \equiv u_2(x) = x^{2\rho+1} e^{-x^2/4}\, {}_pM_q^{\tau}\left[(g_p);(h_q);ax^{2\mu}\right] I_{P_i,Q_i:R}^{M,N}\left[zx^{2\sigma}\right]. \qquad (5.6)$$

Combining (5.2) and (5.5) and making the use of integral (3.1), we derive;

$$C_{2n} = \frac{2^{\rho+n}}{(2n)!\,\sqrt{(\pi)}} \sum_{k=0}^{\infty} M(k) I_{P_i+2,Q_i+1:R}^{M,N+2}\left[z2^{\sigma}\left|\begin{matrix}\left(\frac{1}{2}-\rho-\mu k,\sigma\right),(-\rho-\mu k,\sigma),\ldots,\ldots \\ \ldots,\ldots,(n-\rho-\mu k,\sigma)\end{matrix}\right.\right]. \qquad (5.7)$$

Similarly from (5.3), (5.6) and (3.3), we obtain;

$$C_{2n+1} = \frac{2^{\rho+n+1}}{(2n+1)!\,\sqrt{(\pi)}} \sum_{k=0}^{\infty} M(k) I_{P_i+2,Q_i+1:R}^{M,N+2}\left[z2^{\sigma}\left|\begin{matrix}\left(-\frac{1}{2}-\rho-\mu k,\sigma\right),(-\rho-\mu k,\sigma),\ldots,\ldots \\ \ldots,\ldots,(n-\rho-\mu k,\sigma)\end{matrix}\right.\right]. \qquad (5.8)$$

Putting the value of C_{2n} from (5.7) in (5.1) and the value of C_{2n+1} from (5.8) in (5.3), we get the following required solution of the problem respectively:

$$u_1(x,t) = \frac{2^{\rho}}{\sqrt{(\pi)}} \sum_{k,n=0}^{\infty} \left(\frac{2^n}{(2n)!}\right) e^{-2knt - x^2/4}\, M(k)$$

$$\bullet\, I^{M,N+2}_{P_i+2,Q_i+1:R}\left[z2^{\sigma}\left|\begin{matrix}\left(\frac{1}{2}-\rho-\mu k,\sigma\right),(-\rho-\mu k,\sigma),\dots,\dots\\ \dots,\dots,(n-\rho-\mu k,\sigma)\end{matrix}\right.\right]He_{2n}(x) \qquad (5.9)$$

and

$$u_2(x,t)=\frac{2^{\rho+1}}{\sqrt{(\pi)}}\sum_{k,n=0}^{\infty}\left(\frac{2^n}{(2n+1)!}\right)e^{-k(2n+1)t-x^2/4}\,M(k)$$

$$\bullet\, I^{M,N+2}_{P_i+2,Q_i+1:R}\left[z2^{\sigma}\left|\begin{matrix}\left(-\frac{1}{2}-\rho-\mu k,\sigma\right),(-\rho-\mu k,\sigma),\dots,\dots\\ \dots,\dots,(n-\rho-\mu k,\sigma)\end{matrix}\right.\right]He_{2n+1}(x). \qquad (5.10)$$

Special Cases:

i. On putting $\tau = 1$, the M-series reduces to generalized hypergeometric function ${}_pF_q(x)$ and we get the solution of (4.1) in the product of Chebyshev Hermite polynomials, generalized hypergeometric function and I-function.

ii. On choosing$p = q = 0$, i.e. no upper and lower parameters the the series reduced to the Mittag-Leffler function.

iii. On reducing M-series into unity and I-function into Fox's H-function we get the solution of (4.1) in terms of H-function, which are given by Bajpai [1, p.33 (5.7) & (5.8)].

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