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Fixed points on two complete D- metric spaces

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ABSTRACT: The measures of nearness between two or more objects with respect to a specific characteristic are parameter of the nearness. In D-metric space thereby which it has been possible to determine the geometrical nearness, generalized metric D is a slight variant of two metric. Fisher [4 & 5] obtained related fixed point theorems on two complete metric spaces. Inspired by his work, in this paper we prove a related fixed point theorem on two complete D-metric spaces.

Keywords: D –metric space; complete metric space and fixed point.

INTRODUCTION: Gähler [6 & 7] introduced the concept of 2-metic and Dhage[1] introduced the concept of generalized metric space (D-metric space) and proved several fixed point theorems in this space. Further, Rhoades[9], Raju Rao and Bhishma Rao[8], Veerapandi and Chandrashekhar Rao[11], Dhage, Pathan and Rhoades[3], Singh and Sharma[10] etc. also used this concept in a fixed point frame work.

PRELIMINARIES

DEFINITION 1: Let X be a non-empty set. *A generalized metric (or D-metric)* on X is a function from X × X × X to the set of real numbers satisfies the following conditions:

(D-1) D(x, y, z) ≥ 0 and equality holds if and only if x = y = z,

(D-2) D(x, y, z) = D(y, x, z) = ……. .. (Symmetry),

(D-3) D(x, y, z) ≤ D(x, y, a) + D(x, a, z) + D(a, y, z) (Rectangle inequality), for all x, y, z, a in X.

The pair (X, D) is called *generalized metric space (or D-metric space)*.

The following functions given by Dhage [1] defined on X × X × X → R, where X is a non-empty set, as

(i) D(x, y, z) = max {d(x, y), d(y, z), d(z, x)},
(ii) D(x, y, z) = d(x, y) + d(y, z) + d(z, x)
for all x, y, z in X, are *D-metrics*, where d is ordinary metric on X.

DEFINITION 2: A sequence $\{x_n\}$ of points in a D-metric space X is said to be *D-convergent* and converges to a point x ∈ X, if for ε > 0 there exists $n_0 \in$ N such that
$D(x_m, x_n, x) < \varepsilon$, for all m, n ≥ n_0.

DEFINITION 3: A sequence $\{x_n\}$ of points in a D-metric space X is said to be *D-Cauchy*, if for ε > 0 there exists $n_0 \in$ N such that
$D(x_m, x_n, x_p) < \varepsilon$, for all m > n, p ≥ n_0.

DEFINITION 4: A D-metric space (X, D) is said to be *complete* if every D-Cauchy sequence in X converges to a point in X.

DEFINITION 5: A self-map T of a D-metric space X is said to be *continuous* if $Tx_n \to Tx$, whenever $x_n \to x$.

Fisher [4 & 5] obtained related fixed point theorems on two complete metric spaces. Inspired by his work, we prove a related fixed point theorem on two complete D-metric spaces. Before stating our results, we mention a lemma given by Dhage[2] which will be required in the sequel.

LEMMA 1: (D-Cauchy Principle **).** Let $\{y_n\}$ be a bounded sequence in a D-metric space X with D-bound k satisfying
$D(y_n, y_{n+1}, y_m) \le \lambda^n k$
for all m > n ∈ N, 0 ≤ λ < 1. Then $\{y_n\}$ is D-Cauchy.

MAIN RESULTS:

We prove the following related theorem:

THEOREM: Let (X, D_1) and (Y, D_2) be bounded and complete D-metric spaces. If T is a continuous mapping of X into Y and S is a mapping of Y into X satisfying the inequalities

(1) D_1^2(STx, STx', STx") ≤ α max {D_1(x, STx, STx')
D_1(x', STx', STx"),
D_1 (x, x', STx') D_1(x', x", STx"),

$D_1(x, x', x'')\, D_1(STx, STx', STx'')$,
$D_1(x', STx', STx'')\, D_2(Tx, Tx', Tx'')\}$,

(2) $D_2^2(TSy, TSy', TSy'') \le \alpha \max\{D_2(y, TSy, TSy')\, D_2(y', TSy', TSy'')$,
$D_1(y, y', STy')\, D_2(y', y'', TSy'')$,
$D_2(y, y', y'')\, D_2(TSy, TSy', TSy'')$,
$D_2(y', TSy', TSy'')\, D_1(Sy, Sy', Sy'')\}$,

for all x, x', x'' in X and y, y', y'' in Y, where $0 \le \alpha < 1$, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, Tz = w and Sw = z.

PROOF: Let x be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by $x = x_0$, $(ST)^n x = x_n$, $T(ST)^{n-1}x = y_n$ for n = 1, 2, ….. . By inequality (1), we have

$D_1^2(x_n, x_{n+1}, x_{n+2}) = D_1^2(STx_{n-1}, STx_n, STx_{n+1})$
$\le \alpha \max\{D_1(x_{n-1}, STx_{n-1}, STx_n)\, D_1(x_n, STx_n, STx_{n+1})$,
$D_1(x_{n-1}, x_n, STx_n)\, D_1(x_n, x_{n+1}, STx_{n+1})$,
$D_1(x_{n-1}, x_n, x_{n+1})\, D_1(STx_{n-1}, STx_n, STx_{n+1})$,
$D_1(x_n, STx_n, STx_{n+1})\, D_2(Tx_{n-1}, Tx_n, x_{n+1})\}$
$= \alpha \max\{D_1(x_{n-1}, x_n, x_{n+1})\, D_1(x_n, x_{n+1}, x_{n+2})$,
$D_1(x_{n-1}, x_n, x_{n+1})\, D_1(x_n, x_{n+1}, x_{n+2})$,
$D_1(x_{n-1}, x_n, x_{n+1})\, D_1(x_n, x_{n+1}, x_{n+2})$,
$D_1(x_n, x_{n+1}, x_{n+2})\, D_2(y_n, y_{n+1}, y_{n+2})\}$

which implies that

$D_1(x_n, x_{n+1}, x_{n+2}) \le \alpha \max\{D_1(x_{n-1}, x_n, x_{n+1}), D_2(y_n, y_{n+1}, y_{n+2})\}$.

Now, there are two cases:

Case I (a): If $\max\{D_1(x_{n-1}, x_n, x_{n+1}), D_2(y_n, y_{n+1}, y_{n+2})\} = D_1(x_{n-1}, x_n, x_{n+1})$, then $D_1(x_n, x_{n+1}, x_{n+2}) \le \alpha D_1(x_{n-1}, x_n, x_{n+1})$ and it now follows by induction that
$D_1(x_n, x_{n+1}, x_{n+2}) \le \alpha^n D_1(x_0, x_1, x_2)$.

Case I(b): If $\max\{D_1(x_{n-1}, x_n, x_{n+1}), D_2(y_n, y_{n+1}, y_{n+2})\} = D_1(y_n, y_{n+1}, y_{n+2})$, then

(3) $D_1(x_n, x_{n+1}, x_{n+2}) \le \alpha D_2(y_n, y_{n+1}, y_{n+2})$.

By (2), we have

$D_2^2(y_n, y_{n+1}, y_{n+2}) = D_2^2(TSy_{n-1}, TSy_n, TSy_{n+1})$
$\le \alpha \max\{D_2(y_{n-1}, TSy_{n-1}, TSy_n)\, D_2(y_n, TSy_n, TSy_{n+1})$,
$D_2(y_{n-1}, y_n, TSy_n)\, D_2(y_n, y_{n+1}, TSy_{n+1})$,
$D_2(y_{n-1}, y_n, y_{n+1})\, D_2(TSy_{n-1}, TSy_n, TSy_{n+1})$,
$D_2(y_n, TSy_n, TSy_{n+1})\, D_1(Sy_{n-1}, Sy_n, Sy_{n+1})\}$.
$= \alpha \max\{D_2(y_{n-1}, y_n, y_{n+1})\, D_2(y_n, y_{n+1}, y_{n+2})$,
$D_2(y_{n-1}, y_n, y_{n+1})\, D_2(y_n, y_{n+1}, y_{n+2})$,
$D_2(y_{n-1}, y_n, y_{n+1})\, D_2(y_n, y_{n+1}, y_{n+2})$,
$D_2(y_n, y_{n+1}, y_{n+2})\, D_1(x_{n-1}, x_n, x_{n+1})\}$

which implies that

$D_2(y_n, y_{n+1}, y_{n+2}) \le \alpha \max\{D_2(y_{n-1}, y_n, y_{n+1}), D_1(x_{n-1}, x_n, x_{n+1})\}$.

Case II(a): If $\max D_2(y_{n-1}, y_n, y_{n+1}), D_1(x_{n-1}, x_n, x_{n+1})\} = D_2(y_{n-1}, y_n, y_{n+1})$,

Then; $D_2(y_n, y_{n+1}, y_{n+2} \le \alpha D_2(y_{n-1}, y_n, y_{n+1})$ and it follows by induction that

$D_2(y_n, y_{n+1}, y_{n+2}) \le \alpha^{n-1} D_2(y_1, y_2, y_3)$.

Case II(b): If $\max\{D_2(y_{n-1}, y_n, y_{n+1}), D_1(x_{n-1}, x_n, x_{n+1})\} = D_1(x_{n-1}, x_n, x_{n+1})$, then

(4) $D_2(y_n, y_{n+1}, y_{n+2}) \le \alpha D_1(x_{n-1}, x_n, x_{x+1})$.

From (3) and (4), we get

$D_1(x_n, x_{n+1}, x_{x+2}) \le \alpha^2 D_1(x_{n-1}, x_n, x_{n+1})$.

By induction, we again have

$D_1(x_n, x_{n+1}, x_{x+2}) \le \alpha^{2n} D_1(x_0, x_1, x_2)$.

Letting n tend to infinity in all cases, it now follows by Lemma 1 that $\{x_n\}$ is a D-Cauchy sequence with a limit z in X and $\{y_n\}$ is a D-Cauchy sequence with a limit w in Y.

Using continuity of T, we now have

$$w = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} Tx_n = Tz .$$

Further, using (1), we get

$D_1^2(STz, x_n, x_{n+1}) = D_1^2(STz, STx_{n-1}, STx_n)$
$\le \alpha \max\{D_1(z, STz, STx_{n-1})\, D_1(x_{n-1}, STx_{n-1}, STx_n)$,
$D_1(z., x_{n-1}, STx_{n-1})\, D_1(x_{n-1}, x_n, STx_n)$,
$D_1(z, x_{n-1}, x_n)\, D_1(STz, STx_{n-1}, STx_n)$,
$D_1(x_{n-1}, STx_{n-1}, STx_n)\, D_2(Tz, Tx_{n-1}, Tx_n)\}$
$= \alpha \max\{D_1(z, STz, x_n)\, D_1(x_{n-1}, x_n, x_{n+1})$,
$D_1(z., x_{n-1}, x_n)\, D_1(x_{n-1}, x_n, x_{n+1})$,
$D_1(z, x_{n-1}, x_n)\, D_1(STz, x_n, x_{n+1})$,
$D_1(x_{n-1}, x_n, x_{n+1})\, D_2(Tz, y_n, y_{n+1})\}$

Letting n tend to infinity, we obtain

$D_1^2(STz, z, z) \le \alpha \max\{D_1(z, STz, z)\, D_1(z, z, z), D_1(z, z, z)\, D_1(z, z, z)$,
$D_1(z, z, z,)\, D_1(STz, z, z), D_1(z, z, z)\, D_2(Tz, w, w)\}$
$= 0$

which yields

$$STz = z = Sw.$$

Similarly, applying (2), we have

$D_2^2(TSw, y_n, y_{n+1}) = D_2^2(TSw, TSy_{n-1}, TSy_n)$
$\le \alpha \max\{D_2(w, TSw, TSy_{n-1})\, D_2(y_{n-1}, TSy_{n-1}, TSy_n)$,
$D_2(w, y_{n-1}, TSy_{n-1})\, D_2(y_{n-1}, y_n, TSy_n)$,
$D_2(w, y_{n-1}, y_n)\, D_2(TSw, TSy_{n-1}, TSy_n)$,
$D_2(y_{n-1}, TSy_{n-1}, TSy_n)\, D_1(Sw, Sy_{n-1}, Sy_n)\}$
$= \alpha \max\{D_2(w, TSw, y_n)\, D_2(y_{n-1}, y_n, y_{n+1})$,
$D_2(w, y_{n-1}, y_n)\, D_2(y_{n-1}, y_n, y_{n+1})$,
$D_2(w, y_{n-1}, y_n)\, D_2(TSw, y_n, y_{n+1})$,

$D_2(y_{n-1}, y_n, y_{n+1})\, D_1(Sw, x_{n-1}, x_n)\}$

Letting n tend to infinity, we simple get

$D_2^2(TSw, w, w) \le 0$

which implies

$$TSw = w = Tz.$$

Hence z is a fixed point of ST and w is a fixed point of TS.

To prove uniqueness of z, suppose that ST has a second fixed point z'. Then by (1), we have

$D_1^2(z', z', z) = D_1^2(STz', STz', STz)$
$\le \alpha \max \{D_1(z', STz', STz')\, D_1(z', STz', STz),$
$D_1(z', z', STz')\, D_1(z', z, STz),$
$D_1(z', z', z)\, D_1(STz', STz', STz),$
$D_1(z', STz', STz)\, D_2(Tz', Tz', Tz)\}$
$= \alpha \max \{D_1(z', z', z')\, D_1(z', z', z),$
$D_1(z', z', z')\, D_1(z', z, z),$
$D_1(z', z', z)\, D_1(z', z', z),$
$D_1(z', z', z)\, D_2(Tz', Tz', Tz)\}$
$= \alpha \max \{D_1^2(z', z', z), D_1(z', z', z)\, D_2(Tz', Tz', Tz)\}$

which simply yields

(5) $D_1(z', z', z) \le \alpha D_2(Tz', Tz', Tz).$

Now, using (2), we get

$D_2^2(Tz', Tz', Tz) = D_2^2(TSTz', TSTz', TSTz)$
$\le \alpha \max \{D_2(Tz', TSTz', TSTz')\, D_2(Tz', TSTz', TSTz),$
$D_2(Tz', Tz', TSTz')\, D_2(Tz', Tz, TSTz),$
$D_2(Tz', Tz', Tz)\, D_2(TSTz', TSTz', TSTz),$
$D_2(Tz', TSTz', TSTz)\, D_1(STz', STz', STz)\}$
$= \alpha \max \{D_2(Tz', Tz', Tz')\, D_2(Tz', Tz', Tz),$
$D_2(Tz', Tz', Tz')\, D_2(Tz', Tz, Tz),$
$D_2(Tz', Tz', Tz)\, D_2(Tz', Tz', Tz),$
$D_2(Tz', Tz', Tz)\, D_1(z', z', z)\}$
$\le \alpha \max \{D_2^2(Tz', Tz', Tz), D_2(Tz', Tz', Tz), D_1(z', z', z)\}$

which implies

(6) $D_2(Tz', Tz', Tz) \le \alpha D_1(z', z', z)$

From (5) and (6), we have

$D_1(z', z', z) \le \alpha^2 D_1(z', z', z) < D_1(z', z', z)$

a contradiction. Therefore z = z'.

Similarly, we can show that w is a unique fixed point of TS. This completes the proof of Theorem 1.

On taking $(X, D_1) = (Y, D_2) = (X, D)$ in Theorem 1, we immediately have

COROLLARY 1: Let (X, D) be a bounded and complete D-metric. Let S and T be self-mappings of X satisfying the inequalities

(7) $D^2(STx, STx', STx'') \le \alpha \max \{D(x, STx, STx')\, D(x', STx', STx''),$
$D(x, x', STx')\, D(x', x'', STx''),$
$D(x', x', x'')\, D(STx, STx', STx''),$
$D(x', STx', STx'')\, D(Tx, Tx', Tx'')\}$

(8) $D_2(TSy, TSy', TSy'') \le \alpha \max \{D(y, TSy, TSy')\, D(y', TSy', TSy''),$
$D(y, y', TSy')\, D(y', y'', TSy''),$
$D(y, y', y'')\, D(TSy, TSy', TSy''),$
$D(y', TSy', TSy'')\, D(Sy, Sy', Sy'')\}$

for all x, x', x'', y, y', y'' in X, where $0 \le \alpha < 1$ and if T is continuous, then ST has a unique fixed point z and TS has a unique fixed point w in X. Further, Tz = w and Sw = z and if z = w, then z is the unique common fixed point of S and T.

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