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Introduction of I- Function in Mechanical Oscillating System using Inductance

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ABSTRACT: In the case of mechanical oscillating system when the I- function is expressed in generalized Hyper-geometric series then the non-linear differential equation formed by free vibration duo to attachment of mass to a spring is transformed into-

$$M\ddot{x} + Kx + \delta x^3 = 0$$

Keywords: Jacobi Polynomials; Non-linear differential equation; approximate solution; Saxena's I-function.

INTRODUCTION: Different branches of engineering and physics, is also concerned with the study of vibrating system problems which requires solving the non-linear differential equations by the method of approximations. The non-linear differential equations with periodic coefficients arise in certain physical problems, like Malde's experiment on the vibration of thread or simple pendulum with oscillating support. In the present paper, author presents the analysis of a resistance less circuit containing a non-linear inductance under the effect of external periodic force of general nature.

Many researchers, e.g. Denman and Liu[1], Grade[3], Saxena & Kushwaha[9], Khan & Verma[4] and Mishra & Shrivastava[5] have accomplished the linearization of the ordinary linear differential equations by approximating the non-linear torque by orthogonal polynomials such as ultra - spherical and Jacobi polynomials.

Following Saxena & Kushwaha[9] and extending the work of Mishra & Shrivastava, we make use of Jacobi -polynomials to obtain the linear amplitude-dependent approximate solution of the non-linear differential equation of general type

$$\ddot{M}x + \omega I^{m,n}_{p_i,q_i,r}\left[\mu\left(1+\frac{x}{A}\right)^{\sigma}\right] = MNF(t) \qquad (1.1)$$

The general initial conditions in which (1.1) can be solved may be taken as $x = A(A-1)$ and $\dot{x} = 0$ at $t = 0$, where $A(A-1)$ is the amplitude of the motion and $I^{m,n}_{p_i,q_i,r}(\cdot)$ is the well known I-function introduced by Saxena[7], will be represented and defined as

$$I^{m,n}_{p_i,q_i,r}\left[x\left|\begin{matrix}\left[(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\right]\\ \left[(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\right]\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_{\mathcal{L}}\theta(s)x^s ds \qquad (1.2)$$

Where:

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\ \prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right]}. \qquad (1.3)$$

For the nature of contour L in (1.2), the convergence, existence conditions and other details of the I-function, one can refer to[10].

Some applications to problems of dynamics have also been shown at the end of the paper. Free oscillations of certain non-linear systems found in Pipes[6] have been discussed. Also the results obtained in Pipes[6] have been improved by using Jacobi polynomial approximation for $\sin x$, instead of assuming $sin\ x$ equal to x, in the differential equations:

$$\ddot{M}x = -\frac{g}{l}\sin x + 2\omega\cos\lambda\dot{y}, \qquad (1.4)$$

$$\ddot{M}y = -\frac{g}{l}\sin y - 2\omega\cos\lambda\dot{x}, \qquad (1.5)$$

$$\ddot{M}x = -\omega^2\sin x + \alpha(\dot{x})^2. \qquad (1.6)$$

PROPER INTEGRAL: The following integral will be required in our present investigation-

$$\int_0^t x^{\rho-1}(t-x)^{\beta-1}{}_UF_V\left\{\begin{matrix}A_U\\B_V\end{matrix};\lambda x^{\nu}(t-x)^{\mu}\right\} I^{m,n}_{p_i,q_i,r}\left[zx^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$

$$= t^{\rho+\beta-1}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{U}(A_j;k)\lambda^k\Gamma(\beta+\mu k)}{\prod_{j=1}^{V}(B_j;k)(k)!}\,t^{(\mu+\nu)k}$$

$$\bullet\, I^{m,n+1}_{p_i+1,q_i+1,r}\left[zt^{\sigma}\left|\begin{matrix}(1-\rho-\nu k,\sigma), & (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}, & (1-\rho-\beta-\nu k-\mu k,\sigma)\end{matrix}\right.\right], \quad (2.1)$$

Provided:

i. $\nu\ and\ \mu$ are non-negative integers such that $\nu+\ \mu\geq 1$.

ii. $Re(\rho)+\sigma\ \min_{1\leq j\leq m}\left[Re\left(\frac{b_j}{\beta_j}\right)\right]>0\ and\ Re(\beta)>0, (j=1,\cdots,m)$,

iii. The above integral holds if $U\leq V(or\ U=V+1\ and\ |\lambda|<1)$, no one of the denominator parameters $B_1,\cdots,B_V$ is zero or a negative integer, and

iv. $A_i>0, B_i\leq 0, |arg\,z|<\frac{\pi}{2}A_i, \forall\ i=1,2,\cdots r$, where A_i and B_i are given by

$$A_i=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_{ji}+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_{ji}, \forall\ i=1,2,\cdots r$$

and

$$B_i=\frac{1}{2}(p_i-q_i)+\sum_{j=1}^{q_i}b_{ji}-\sum_{j=1}^{p_i}a_{ji},\ \forall\ i=1,2,\cdots r.$$

The integral (2.1) is the special case of integral [8, p. 63 (eq. 4.4.2)] and we can established by replacing the generalized hypergeometric function ${}_UF_V$ in series form and the I-function on the left-hand side by its equivalent contour integral as given in (1.2), changing the order of integrations which is justifiable due to the absolute convergence of the integrals, evaluating the inner integral with the help of [2], we get the desired result (2.1).

Deductions:

i. Taking $\lambda=t=V=\mu=1, U=2, A_1=a, A_2=b, B_1=\beta, \nu=0$ in (2.1), replacing the I-function on the right- hand side by its equivalent contour integral as given in (1.2), then changing the order of summation and integration and evaluating the inner summation with the help of Gauss theorem[2], we have:

$$\int_0^1 x^{\rho-1}(1-x)^{\beta-1}{}_2F_1\left\{\begin{matrix}a,b\\\beta\end{matrix};(1-x)\right\} I^{m,n}_{p_i,q_i,r}\left[zx^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$

$$=\Gamma(\beta)I^{m,n+2}_{p_i+2,q_i+2,r}\left[z\left|\begin{matrix}(1-\rho,\sigma),(1+a+b-\rho-\beta,\sigma), & (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}, & (1+a-\rho-\beta,\sigma)(1+b-\rho-\beta,\sigma)\end{matrix}\right.\right], \quad (2.2)$$

Provided that $Re(\rho+\beta-a-b)>0$, $Re(\beta)>0$, and $Re(\rho)+\sigma\ \min_{1\leq j\leq m}\left[Re\left(\frac{b_j}{\beta_j}\right)\right]>0$

and $A_i>0, B_i\leq 0, |arg\,z|<\frac{\pi}{2}A_i, \forall\ i=1,2,\cdots r$, where A_i and B_i are given with equation (2.1).

ii. Replacing $a\ by-k, b\ by\ \alpha+\beta+k+1, \beta\ by\ 1+\alpha,\ z\ by\ \mu 2^{\sigma},\ x\ by\ \frac{1+x}{2}$, ρ by $1+\beta$ and expressing ${}_2F_1$ in Jacobi polynomials in (2.2), we obtain:

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x)\,I^{m,n}_{p_i,q_i,r}\left[\mu(1+x)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx,$$

$$=\frac{2^{\alpha+\beta+1}}{(k)!}\Gamma(\alpha+k+1)\,I^{m,n+2}_{p_i+2,q_i+2,r}\left[\mu 2^{\sigma}\left|\begin{matrix}(-\beta,\sigma),(0,\sigma), & (a_j,\alpha_j)_{1,n},(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m},(b_{ji},\beta_{ji})_{m+1,q_i}, & (-\alpha-\beta-k-1,\sigma)(k,\sigma)\end{matrix}\right.\right]$$

(2.3)

which exists under the conditions given with (2.1) along with $Re\left(1+\beta+\sigma\frac{b_j}{\beta_j}\right)>0$ $(j=1,\cdots,m);\ and\ Re(1+\alpha)>0$ and $A_i>0, B_i\leq 0, |arg\,\mu 2^{\sigma}|<\frac{\pi}{2}A_i, \forall\, i=1,2,\cdots r$, where A_i and B_i are given with equation (2.1)

Mechanical Oscillations and Linear Approximation:

Mechanical Oscillations can be expressed as the sets of polynomial orthogonal in the interval $(-A,A)$ with respect to the weight function $\left\{1-\frac{x}{A}\right\}^{\alpha}\left\{1+\frac{x}{A}\right\}^{\beta}$. This gives rise to the polynomials $P_k^{(\alpha,\beta)}\left(\frac{x}{A}\right)$.

For a function $f(x)$, which can be expanded in terms of the Jacobi polynomials in the interval $(-A,A)$, we obtain:

$$f(x)=\sum_{k=0}^{\infty}a_k^{(\alpha,\beta)}P_k^{(\alpha,\beta)}\left(\frac{x}{A}\right) \tag{3.1}$$

where, the coefficients $a_k^{(\alpha,\beta)}$ are given by

$$a_k^{(\alpha,\beta)}=\frac{\int_{-1}^{1}f(Ax)(1-x)^{\alpha}(1+x)^{\beta}P_k^{(\alpha,\beta)}(x)dx}{\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[P_k^{(\alpha,\beta)}(x)\right]^2dx} \tag{3.2}$$

If the series (3.1) is truncated after the second term, we obtain a linear approximation

$$f_*(x)=a_0^{(\alpha,\beta)}P_0^{(\alpha,\beta)}\left(\frac{x}{A}\right)+a_1^{(\alpha,\beta)}P_1^{(\alpha,\beta)}\left(\frac{x}{A}\right) \tag{3.3}$$

where, star indicates approximation.

Mechanical Oscillations and Non-linear Differential Equation:

Here we solve the non-linear differential equation:

$$M\ddot{x}+f(x)=MNF(t) \tag{4.1}$$

where,

$$f(x)=\omega I_{p_i,q_i,r}^{m,n}\left[\mu\left(1+\frac{x}{A}\right)^{\sigma}\right]$$

by approximating $f(x)$ in the interval $(-A,A)$, with the help of linear Jacobi polynomials.

From (3.3) we have:

$$\begin{aligned} f_*(x)&=\omega I_{p_i,q_i,r}^{m,n}\left[\mu\left(1+\frac{x}{A}\right)^{\sigma}\right]_*\\ &=a_0^{(\alpha,\beta)}P_0^{(\alpha,\beta)}\left(\frac{x}{A}\right)+a_1^{(\alpha,\beta)}P_1^{(\alpha,\beta)}\left(\frac{x}{A}\right)\\ &=a_0^{(\alpha,\beta)}+a_1^{(\alpha,\beta)}\left[\frac{\alpha-\beta}{2}+\frac{\alpha+\beta+2}{2}\frac{x}{A}\right] \end{aligned} \tag{4.2}$$

where,

$$a_0^{(\alpha,\beta)}=\frac{\int_{-1}^{1}\omega I_{p_i,q_i,r}^{m,n}[\mu(1+x)^{\sigma}](1-x)^{\alpha}(1+x)^{\beta}P_0^{(\alpha,\beta)}(x)dx}{\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[P_0^{(\alpha,\beta)}(x)\right]^2dx} \tag{4.3}$$

and

$$a_1^{(\alpha,\beta)}=\frac{\int_{-1}^{1}\omega I_{p_i,q_i,r}^{m,n}[\mu(1+x)^{\sigma}](1-x)^{\alpha}(1+x)^{\beta}P_1^{(\alpha,\beta)}(x)dx}{\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\left[P_1^{(\alpha,\beta)}(x)\right]^2dx} \tag{4.4}$$

using the result (2.3) we find that:

$$a_0^{(\alpha,\beta)}=\frac{\omega\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)}I_{p_i+1,q_i+1,r}^{m,n+1}\left[\mu 2^{\sigma}\left|\begin{matrix}(-\beta,\sigma), & \cdots\\ \cdots,(-\alpha-\beta-1,\sigma)\end{matrix}\right.\right] \tag{4.5}$$

and

$$a_1^{(\alpha,\beta)}=\frac{\omega(\alpha+\beta+3)\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)}I_{p_i+2,q_i+2,r}^{m,n+2}\left[\mu 2^{\sigma}\left|\begin{matrix}(-\beta,\sigma),(0,\sigma), & \cdots\\ \cdots,(-\alpha-\beta-1,\sigma),(1,\sigma)\end{matrix}\right.\right] \tag{4.6}$$

Replacing $f(x)$ by its approximation $f_*(x)$, the equation (4.1) transforms into

$$\ddot{M}x + a_0^{(\alpha,\beta)} + a_1^{(\alpha,\beta)}\left[\frac{\alpha-\beta}{2} + \frac{\alpha+\beta+2}{2}\,\frac{x}{A}\right] = MNF(t) \tag{4.7}$$

or

$$\ddot{M}x + \gamma^2 x = -\frac{(\alpha-\beta)A}{(\alpha+\beta+2)}\left[\gamma^2 - \gamma_1^2\right] + MNF(t) \tag{4.8}$$

where,

$$\gamma^2 = \frac{(\alpha+\beta+2)}{2A}a_1^{(\alpha,\beta)} \qquad and \qquad \gamma_1^2 = \frac{(\alpha+\beta+2)}{(\beta-\alpha)A}a_0^{(\alpha,\beta)} \tag{4.9}$$

The value of $a_0^{(\alpha,\beta)}$ and $a_1^{(\alpha,\beta)}$ are given by (4.5) and (4.6) respectively.
The approximate general solution of (4.8) subject to the initial conditions $x = A(A-1), \dot{x} = 0$ at $t = 0$ is given by:

$$x_* = \left[A(A-1) + \frac{(\alpha-\beta)A}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_1^2}{\gamma^2}\right)\right]\cos\gamma t - \frac{(\alpha-\beta)A}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_1^2}{\gamma^2}\right) + \frac{N}{\gamma}\int_0^t F(u)\sin\gamma(t-u)du, \tag{4.10}$$

Which is most general approximate solution and on giving different values to $F(t)$ in (4.1), one can find approximate general solution corresponding to $F(t)$. The electrical analogue of (4.1) can be given by a circuit consisting of a linear inductor L in series with a non-linear inductance and a harmonic potential$MNF(t)$. If x is the charge separation of the plates of the inductance, the differential equation of the circuit as introduced by Garde[3] is given by:

$$L\ddot{x} + f(x) = NF(t) \tag{4.11}$$

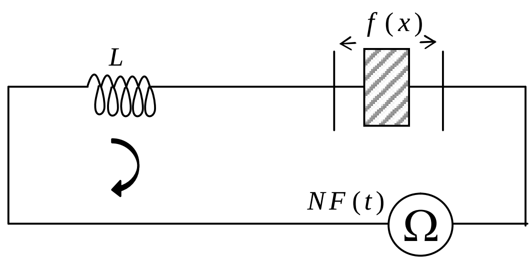

Figure 1: Resistanceless circuit containing a non-linear inductance

Where $f(x)$ is potential drop across the non-linear capacitor. Generally $f(x)$ is in the form of a curve between voltage and charge known as saturation curve.

In (4.1) choosing $\omega = \frac{g}{l}\sqrt{\pi},\ m = 1,\ n = p_i = b_2 = 0, b_1 = \frac{1}{2}, q_i = \sigma = 2, \mu = \frac{1}{4},\ x = A(x-1),\ \beta_1 = \beta_2 = r = 1$ converting I-function into sine form and replacing $N\ by\ 2\omega\cos\lambda, F(t)\ by\ \frac{dy}{dt}$, one obtains differential equation (1.4) and hence from (4.2) we get the approximation for sine non-linearity.
Making the corresponding changes for the above choice the values of γ and γ_1 in (4.9), are transformed to γ_2^2 and γ_3^2 respectively, where;

$$\gamma_2^2 = \frac{g\sqrt{\pi}\Gamma(\alpha+\beta+4)}{2Al\,\Gamma(\beta+2)}\, I_{2,4;1}^{1,2}\left[1\left|\begin{matrix}(-\beta,2),(0,1)\\(1/2,1),(0,1),(-\alpha-\beta-2,2),(1,2)\end{matrix}\right.\right] \tag{5.1}$$

and

$$\gamma_3^2 = \frac{g\sqrt{\pi}\Gamma(\alpha+\beta+3)}{Al(\beta-\alpha)\,\Gamma(\beta+1)}\, I_{1,3;1}^{1,1}\left[1\left|\begin{matrix}(-\beta,2)\\(1/2,1),(0,1),(-\alpha-\beta-1,2)\end{matrix}\right.\right] \tag{5.2}$$

Hence with the help of (4.10), the improved approximate solution of (1.4) is given as:

$$x_* = \left[A(A-1) + \frac{(\alpha-\beta)A}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_3^2}{\gamma_2^2}\right)\right]\cos\gamma_2 t - \frac{(\alpha-\beta)A}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_3^2}{\gamma_2^2}\right) + \frac{2\omega\cos\lambda}{\gamma}\int_0^t \frac{dy}{du}\sin\gamma_2(t-u)du, \tag{5.3}$$

subject to the initial conditions $x = A, \dot{x} = 0$ at $t = 0$.
Proceeding on parallel lines as above one may obtain the solution of (1.5) from (4.10). The differential equation (1.6) depicts a kind of Newtonian damped motion for a freely vibrating system whose damping force is proportional to the square of the velocity. In (1.6) substituting the approximate value of sine non-linearity derived from (4.2) on choosing the parameters suitably, we obtain

$$M\ddot{x} + \gamma_2^2 x = -\frac{A(\alpha-\beta)}{(\alpha+\beta+2)}(\gamma_2^2 - \gamma_3^2) + \alpha(\dot{x})^2 \tag{5.4}$$

For the first approximation, the small term $\alpha(\dot{x})^2$ may be neglected and we get the solution

$$x_* = \xi(\cos\gamma_2 t + \eta) - \frac{A(\alpha-\beta)}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_3^2}{\gamma_2^2}\right) \tag{5.5}$$

where, ξ and η are arbitrary constants.
For the second approximation putting this value in the small term $\alpha(\dot{x})^2$ one finds from (5.4)

$$M\ddot{x} + \gamma_2^2 x = -\frac{A(\alpha-\beta)}{(\alpha+\beta+2)}(\gamma_2^2 - \gamma_3^2) + \alpha\xi^2\gamma_2^2 sin^2(\gamma_2 t + \eta)$$

or

$$M\ddot{x} + \gamma_2^2 x = -\frac{A(\alpha-\beta)}{(\alpha+\beta+2)}(\gamma_2^2 - \gamma_3^2) + \alpha\xi^2\gamma_2^2[1 - 2\cos(2\gamma_2 t + 2\eta)] \tag{5.6}$$

Hence the general solution of (5.6) is:

$$x_* = \xi(\cos\gamma_2 t + \eta) - \frac{A(\alpha-\beta)}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_3^2}{\gamma_2^2}\right) + \frac{\alpha\xi^2}{2} + \frac{\alpha\xi^2}{6}\cos(2\gamma_2 t + 2\eta) \tag{5.7}$$

where, the constants ξ and η can be determined with the help of the initial conditions $x = A, \dot{x} = 0$ at $t = 0$

The equation (4.1) represents the forced non-linear oscillations without damping. If we put $N = 0$, the equation (4.1) reduces to

$$\ddot{x} + \omega\, I_{p_i,q_i:r}^{m,n}\left[\mu\left(1 + \frac{x}{A}\right)^{\sigma}\right] = 0, \tag{6.1}$$

which represents the free non-linear oscillations without damping.
Hence from (4.10) the approximate solution of (6.1) subject to the initial conditions $x = A(A-1), \dot{x} = 0$at $t = 0$ is given by:

$$x_* = A(A-1) + \frac{A(\alpha-\beta)}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_1^2}{\gamma^2}\right)\cos\gamma t - \frac{A(\alpha-\beta)}{(\alpha+\beta+2)}\left(1 - \frac{\gamma_1^2}{\gamma^2}\right). \tag{6.2}$$

where γ^2 and γ_1^2 are same as in (4.9).
The approximate period of oscillations is given by

$$T_* = \frac{2\pi}{\gamma} \tag{6.3}$$

In the case of a mechanical oscillating system consisting of a mass attached to a spring, the equation fo the free vibration of such a system is given[6] by

$$M\ddot{x} + Kx + \delta x^3 = 0 \tag{6.4}$$

where, $M\ddot{x}$ is the inertia force of the mass, $Kx + \delta x^3$ is spring force, and x is measured from the position of equilibrium of the mass when the spring is not stressed.
On putting $N = 0, m = r = 1, q_i = \sigma = 2, n = p_i = b_2 = 0,$ $b_1 = \frac{1}{2}, \mu = \frac{1}{4}, x = A(x-1)$ in (1.1) and taking only two terms of the expansion of sine series after expressing I- function in the sine form and finally letting $\frac{\omega}{\sqrt{\pi}} = \frac{K}{M}, -\frac{\omega}{3!\sqrt{\pi}} = \frac{\delta}{M}$, we get (6.4). Now the approximate solution of (6.4) in terms of x and t can be obtained from (4.10) making the corresponding changes in the values of γ and γ_1.

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