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Cone Rectangular Metric Spaces and Banach Fixed Point Theorems

S. C. Ghosh[1*] and A. Shukla[2]

[1] *Department of Mathematics, D. A. V. College, Kanpur (U.P.) INDIA*
[2] *Department of Mathematics, Pratap University, Jaipur, Rajasthan, INDIA*
[*] *Correspondence: E-mail: ghosg_subal_123@yahoo.com & scgmathsdav.csjmu@gmail.com*
Akanksha.sam@gmail.com

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ABSTRACT: In this present paper we have establish some generalized Banach fixed point theorems on complete Cone rectangular metric spaces. Here we have proved fixed point theorems by taking f, g, h are self mappings from X into itself.

Keywords: Cone Metric Spaces; Rectangular Cone Metric Spaces; Complete Cone Metric Spaces; Complete Cone Rectangular Metric Spaces; Cauchy Sequence; Banach Fixe point.

INTRODUCTION: Recently H. L. Guang and Z. Xian introduced a new metric space known as cone metric spaces. Subsequently A. Azam., M. Arshad and I. Beg have given the idea of cone rectangular metric space. We will introduce new results on cone rectangular metric space.

Let E is a real Bench space and P is a subset of E. P is called a cone if and only if it satisfies the following conditions

i) P is closed, non empty and $P \neq \{0\}$

ii) a, b $\in$ R and a, b $\geq$ 0. u, v $\in P \Rightarrow a u + b v \in P$

iii) $u \in P$ and $- u \in P \Rightarrow u = 0$.

Definition: Let X be a nonempty set. Suppose that d is a mapping from $X \times X \rightarrow E$, satisfies,

i) $d(x, y) > 0, \forall x, y \in X$

ii) $d(x, y) = 0$ if and only if $x = y$

iii) $d(x, y) = d(y, x), \forall x, y \in X$

iv) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called cone metric space.

Definition: Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow E$, satisfies

i) $o \leq d(x, y); \forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

ii) $d(x, y) = d(y, x), \forall x, y \in X$.

iii) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y), \forall x, y \in X$ and for all distinct point $w, z \in X - \{x, y\}$ [rectangular property]

Then d is called a cone rectangular metric on X, and (X, d) is called a cone rectangular metric space.

Definition: Let $\{x_n\}$ be a sequence in (X, d) and $x \in$ (X, d). If for every $c \in E$, with $0 \ll c$ there is $n_o \in N$ such that for all $n > n_o$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and converges to x.

i.e. $$\lim_{n \to \infty} x_n = x$$

Definition: A sequence (x_n) is said to be Cauchy in X if for $c \in E$ with $o \ll c$ there is $n_o \in N$ such that for all $n, m > n_o$, $d(x_n, x_m) \ll c$ then $\{x_n\}$ is called Cauchy sequence.

Definition: A cone rectangular metric space is said to be complete cone rectangular metric space if every Cauchy sequence in X is convergent.

RESULTS:

Theorem 1: Let (X, d) is a complete cone rectangular metric space and P is a normal cone with normal constant K. Let f is self mapping from X into itself satisfying,

$$d(fx,fy) \leq \alpha \frac{d(x,y) + d(x,fx)}{2} + \beta\, d(y,fy) \ \forall\, x, y \in X,$$

$\alpha, \beta \in (0, 1)$ and $0 < \frac{\alpha}{1-\beta} < 1$, Then f has an unique fixed point in X.

Proof: Let $x_o \in X$ be an arbitrary point in X. Let us take a sequence $\{x_n\}$ in X such that,

$$x_{n+1} = fx_n = f^{n+1}_{x_o}, \ n \in N \cup \{0\}.$$

Now substituting $x = x_o$ and $y = x_1$ in inequality we obtain:

$$d(fx_o, fx_1) \leq \alpha \frac{d(x_0, x_1) + d(x_0, fx_0)}{2} + \beta\, d(x_1, f x_1)$$

$$\Rightarrow d(x_1, x_2) \leq \alpha \frac{d(x_0, x_1) + d(x_0, x_1)}{2} + \beta\, d(x_1, x_2)$$

$$\Rightarrow d(x_1, x_2) \leq \alpha\, d(x_0, x_1) + \beta\, d(x_1, x_2)$$

$$\Rightarrow (1 - \beta) d(x_1, x_2) \leq \alpha\, d(x_0, x_1)$$

$\Rightarrow d(x_1, x_2) \leq \frac{\alpha}{1-\beta} d(x_0, x_1)$

Let us take $\frac{\alpha}{1-\beta}$ = h then from above inequality we obtain:

$d(x_1, x_2) \leq h\, d(x_0, x_1)$ --------------(1.1)

Again for $x = x_1$ and $y = x_2$ in inequality we get,

$d(fx_1, fx_2) \leq \alpha \frac{d(x_1, x_2) + d(x_1, fx_1)}{2} + \beta\, d(x_1, f x_1)$

$d(x_2, x_3) \leq \alpha \frac{d(x_1, x_2) + d(x_1, x_2)}{2} + \beta\, d(x_2, x_3)$

$\Rightarrow (1-\beta)\, d(x_2, x_3) \leq \alpha\, d(x_1, x_2)$

$\Rightarrow d(x_2, x_3) \leq \frac{\alpha}{1-\beta} d(x_1, x_2)$

$\Rightarrow d(x_2, x_3) \leq h^2 d(x_0, x_1)$ ---------------(1.2)

Continuing the same process we obtain in general for $n \in N$,

$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1$ --------------(1.3)

Now for n > m we can find that,

$d(x_n, x_m) \leq (h^{n-1} + h^{n-2} + h^{n-3} + h^{n-4} + h^{n-5} + \text{-------} h^m)\, d(x_0, x_1)$

$\Rightarrow d(x_n, x_m) \leq \frac{h^m}{1-h} d(x_0, x_1)$ ---------------(1.4)

Taking the normality of Cone, (1.4) gives,

$\| d(x_n, x_m) \| \leq K \| \frac{h^m}{1-h} d(x_0, x_1) \|$

$\Rightarrow \| d(x_n, x_m) \| \leq K \| \frac{h^m}{1-h} \| \| d(x_0, x_1) \|$-----------(1.5)

Which yields,

$\| d(x_n, x_m) \| \to 0$ as $n, m \to \infty$ and $\frac{h^m}{1-h} \to 0$.

Now we will claim that our inequality satisfies the rectangular property for finding the fixed point in X. Because of this we will calculate the following results.

For $y \in X$ we have,

$d(fx, f^2x) \leq \alpha \frac{d(x, fx) + d(x, fx)}{2} + \beta\, d(fx, f^2x)$

$\Rightarrow (1-\beta) d(fx, f^2x) \leq \alpha\, d(x, fx)$

$\Rightarrow d(fx, f^2x) \leq \frac{\alpha}{1-\beta} d(x, fx)$ -----------(1.6)

Let us take $0 < \frac{\alpha}{1-\beta} = h < 1$, then the inequality (1.6) gives,

$\Rightarrow d(fx, f^2x) \leq h\, d(x, fx)$ --------------(1.7)

Again we have,

$d(f^2x, f^3x) \leq \alpha \frac{d(fx, f^2x) + d(fx, f^2x)}{2} + \beta\, d(f^2x, f^3x)$

$\Rightarrow d(f^2x, f^3x) \leq \frac{\alpha}{1-\beta} d(fx, f^2x)$

We have taking $h = \frac{\alpha}{1-\beta}$

$d(f^2x, f^3x) \leq h\, d(fx, f^2x)$

$\Rightarrow d(f^2x, f^3x) \leq h\, hd(x, fx)$

$\Rightarrow d(f^2x, f^3x) \leq h^2 d(x, fx)$ ---------------(1.8)

Now continuing in this same technique we obtain in general for positive integer

$\Rightarrow d(f^n x, f^{n+1} x) \leq h^n d(x, fx)$ ----------------(1.9)

Now from rectangular property we have for $x \in X$,

$d(fx, f^4x) \leq d(fx, f^2x) + d(f^2x, f^3x) + d(f^3x, f^4x)$

$\Rightarrow d(fx, f^4x) \leq hd(x, fx) + h^2 d(fx, f^2x) + h^3 d(f^2x, f^3x)$

$\Rightarrow d(fx, f^4x) \leq \sum_{i=1}^{3} h^i\, d(x, fx)$

Similarly,

$d(f^2y, f^5y) < d(f^2y, f^3y) + d(f^3y, f^4y) + d(f^4y, f^5y)$

$\leq h^2\, d(y, fy) + h^3 d(y, fy) + h^4 d(y, fy)$

$\leq \sum_{i=2}^{4} h^i\, d(y, fy)$ ------------------(1.10)

Thus in general for n > m we obtain from Lemma of [4, 6]

$d(f^n y, f^m y) \leq (h^{n-1} + h^{n-2} + h^{n-3} + --- h^m)\, d(y, fy)$

$\leq \frac{h^m}{1-h} d(y, fy)$

Now for $x_o = y \in X$, we obtain,

$d(f^n x_o, f^m x_o) \leq \frac{h^m}{1-h} d(x_o, fx_o)$

$\Rightarrow d(x_n, x_m) \leq \frac{h^m}{1-h} d(x_o, x_1)$

Applying the normality of cone we obtain,

$\| d(x_n, x_m) \| \leq \frac{h^m}{1-h} K \| d(x_o, x_1) \|$

Which implies that $\| d(x_n, x_m) \| \to 0$

$\Rightarrow d(x_n, x_m) \to 0$

As X is complete cone rectangular metric space then there exists a point x in (X, d) such that, $x_n \to x$ as $n \to \infty$

Now $d(x_n, fx_n) = d(fx_{n+1}, fx_n) \to 0$ as $n, m \to \infty$

Now $d(x_{n+1}, fx_{n+1}) = d(fx_n, fx_{n+1})$

$\leq \alpha \frac{d(x_n, xn+1) + d(x_n, fx_{n+1})}{2} + \beta\, d(x_{n+1}, f x_{n+1})$

$\Rightarrow d(x_{n+1}, fx_{n+1}) \leq \alpha \frac{d(x3_n, xn+1) + d(x_n, x_{n+1})}{2} + \beta d(x_{n+1}, x_{n+2})$

$\Rightarrow d(x_{n+1}, fx_{n+1}) \leq \alpha\, d(x_n, x_{n+1}) + \beta\, d(x_{n+1}, x_{n+2})$

$d(x_{n+1}, f x_{n+1})\ (1-\beta) \leq \alpha\, d(x_n, x_{n+1})$

Now letting $n \to \infty$ we obtain from the above inequality,

$d(x, f x)\ (1-\beta) \leq \alpha\, d(x, x)$

$\Rightarrow d(x, f x)\ (1-\beta) \leq 0$

Now applying the normality of cone we have,

$(1 - 2\alpha)\, K \| d(x, fx) \| \leq 0$

Which is a contradiction as $(1-\beta) \geq 0$ and $d(x, f x) \geq 0$ therefore it is only possible that,

$d(x, f x) = 0$

$\Rightarrow f x = x$ ----------------------(1.11)

Hence x is a common fixed point of f in X. Now we will prove that x is unique. If possible let there exists another fixed point x' of f in (X, d).

$$fx' = x' \quad \text{--------------------(1.12)}$$

Then $d(x, x') = d(fx, fx')$

$$d(x,x') \le \alpha \frac{d(x,x')+d(x,fx')}{2} + \beta\, d(x',fx')$$

$$\Rightarrow d(x,x') \le \alpha \frac{d(x,x')+d(x,x')}{2} + \beta\, d(x',x')$$

$$\Rightarrow d(x,x') \le \alpha\, d(x',x')$$

$$(1-\alpha)\, d(x,x') \le 0$$

Again applying the normality of cone we have,

$$(1-2\alpha)\, K\|d(x,x')\| \le 0$$

Again a contradiction as $(1-\beta) \ge 0$ and $d(x, fx) \ge 0$ therefore it is only possible that,

$$d(x,x') = 0$$

$$\Rightarrow x = x' \quad \text{----------------------(1.13)}$$

Hence x is a unique common fixed point of f in X.

Theorem 2: Let (X, d) is a complete cone rectangular metric space and P is a normal cone with normal constant K. Let f and g are self mapping from X into itself satisfying,

$d(fx,gy) \le \alpha\, d(x,y) + \beta d(x,fx) + \gamma\, d(y,gy)\ \forall\ x,y \in X$ and $\alpha,\beta,\gamma \in (0,1)$ also $(1-\alpha)$, $(1-\beta)$, $(1-\gamma) \in (0,1)$ and $\frac{\alpha+\beta}{1-\gamma} \in (0,1)$Then f an g has unique common fixed point in X.

Proof: Let x_0, y_0 are arbitrary point in X. Let us Consider the sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$x_n = fx_{n-1} \text{ and } x_{n+1} = gx_n.$$

Substituting $x = x_o$ and $y = x_1$ in above inequality we obtain,

$$d(x_1,x_2) = d(fx_0,gx_1)$$

$$\le \alpha\, d(x_0,x_1) + \beta d(x_0,fx_0) + \gamma\, d(x_1,gx_1)$$

$$\Rightarrow d(x_1,x_2) \le \alpha\, d(x_0,x_1) + \beta d(x_0,x_1) + \gamma\, d(x_1,x_2)$$

$$\Rightarrow (1-\gamma)\, d(x_1,x_2) \le (\alpha+\beta) d(x_0,x_1)$$

$$\Rightarrow d(x_1,x_2) \le \frac{\alpha+\beta}{1-\gamma} d(x_0,x_1)$$

$$\Rightarrow d(x_1,x_2) \le h\, d(x_0,x_1)$$

Where $0 < h = \frac{\alpha+\beta}{1-\gamma} < 1$ -------------------(2.1)

Again we have for x_2, x_3 we have,

$$\Rightarrow d(x_2,x_3) \le h^2\, d(x_0,x_1) \quad \text{-------------------(2.2)}$$

Continuing similar method we get,

$$\Rightarrow d(x_n,x_{n+1}) \le h^n\, d(x_0,x_1) \quad \text{-------------------(2.3)}$$

Now for n > m we can find that,

$d(x_n, x_m) \le (h^{n-1} + h^{n-2} + h^{n-3} + h^{n-4} + h^{n-5} + \text{--------} h^m)\ d(x_0, x_1)$

$$\Rightarrow d(x_n, x_m) \le \frac{h^m}{1-h} d(x_0, x_1) \quad \text{------------------(2.4)}$$

Taking the normality of Cone, (1.4) gives,

$$\| d(x_n, x_m)\| \le K \left\| \frac{h^m}{1-h} d(x_0, x_1) \right\|$$

$$\Rightarrow \| d(x_n, x_m) \| \le K \left\| \frac{h^m}{1-h} \right\| \|d(x_0, x_1)\| \quad \text{--------(2.5)}$$

Which yields,

$$\| d(x_n, x_m)\| \to 0 \text{ as n, m} \to \infty \text{ and } \frac{h^m}{1-h} \to 0.$$

Which shows that x_n is a Cauchy sequence.

As we have prove that the inequality of the theorem (1.1) is satisfies the rectangular property of Cone rectangular Metric Spaces, Similarly we can established that the inequality of Theorem (2) also satisfies the rectangular property of Cone rectangular Metric Spaces.

$$\Rightarrow d(x_n, x_m) \le \frac{h^m}{1-h}\, d(x_o, x_1)$$

Applying the normality of cone we obtain,

$$\| d(x_n, x_m) \| \le \frac{h^m}{1-h}\, K\, \| d(x_o, x_1) \|$$

Which implies that $\|d(x_n, x_m)\| \to 0$

$$\Rightarrow d(x_n, x_m) \to 0$$

As X is complete cone rectangular metric space then there exists a point

x in (X, d) such that, $x_n \to x$ as $n \to \infty$

Now $d(x_n, fx_n) = d(fx_{n+1}, fx_n) \to 0$ as n, m $\to \infty$

Now $d(fx_{n-1}, x_{n+1}) = d(fx_{n-1}, gx_{n+1})$

$$\le \alpha\, d(x_{n-1},x_n) + \beta d(x_{n-1},fx_{n-1}) + \gamma\, d(x_{n+1},gx_{n+1})$$

$$\le \alpha\, d(x_{n-1},x_n) + \beta d(x_{n-1},fx_{n-1}) + \gamma\, d(x_{n+1},x_n)$$

Now letting as $n \to \infty$ we have from the above in equality,

$$d(fx, x) \le \alpha\, d(x,x) + \beta d(x,fx) + \gamma\, d(x,x)$$

$$\Rightarrow d(fx, x) \le \beta d(x,fx)$$

$\Rightarrow (1-\beta)\, d(fx, x) \le 0$ which is not possible as $(1-\beta) > 0$ and $d(fx, x) \ge 0$.Thus only possible that $d(fx, x) = 0$ that is implies that $fx = x$. Therefore we have that fx has the fixed x point in X.

Again we have,

$$d(x_n, gx_n) = d(fx_{n-1}, gx_n)$$

$$\le \alpha\, d(x_{n-1},x_n) + \beta d(x_{n-1},fx_{n-1}) + \gamma\, d(x_n,gx_n)$$

$$\le \alpha\, d(x_{n-1},x_n) + \beta d(x_{n-1},x_n) + \gamma\, d(x_{n+1},x_n)$$

Now letting as $n \to \infty$ we have from the above in equality,

$$d(x,gx) \le \alpha\, d(x,x) + \beta d(x,x) + \gamma\, d(x,gx)$$

$$\Rightarrow d(x,gx) \le \gamma\, d(x,gx)$$

$\Rightarrow (1-\gamma)\, d(x, gx) \le 0$ which is not possible as $(1-\gamma) > 0$ and $d(gx, x) \ge 0$. Thus only possible that $d(gx, x) = 0$ that is implies that $gx = x$. Therefore we have that gx has the fixed x point in X.

Hence f has the fixed point x in X and g has the fixed point x in X.

i.e. $fx = x$, $gx = x$ ------------------(2.6)

Lastly we prove that f and g Has the unique fixed point in X.

If possible let there existed another fixed point x' in X.

i.e. $fx' = x'$, $gx' = x'$ -----------------(2.7)

Now we have from the inequality for $x = x$ and $y = x'$,

$d(x, x') = d(fx,gx') \le \alpha\, d(x,x') + \beta d(x,fx) + \gamma\, d(x',gx')$ which gives

$d(x, x') \leq \alpha\, d(x,x') + \beta d(x,x) + \gamma\, d(x',x')$

$\Rightarrow d(x, x') \leq \alpha\, d(x,x')$

$\Rightarrow (1-\alpha)\, d(x, x') \leq 0.$

Again a contradiction and hence it yields

$d(x, x') = 0$

$\Rightarrow x = x$ ---------------------(2.8)

Hence f and g has the unique common fixed point in X.

Theorem 2.1: Let (X, d) is a complete cone rectangular metric space and P is a normal cone with normal constant K. Let f and g are self mapping from X into itself and f is commutes with g and satisfying the condition,

$d(fx,gy) \leq \alpha\, d(gx,gy) + \beta d(fx,gx) + \gamma\, d(fy,gy)$

$\forall x,y \in X$ and $\alpha,\beta,\gamma \in (0,1)$ also $\frac{\alpha+\beta}{1-\gamma}$, $(1-\beta-\gamma)$ and $(1-\alpha) \in (1,0)$ Then f an g are unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Let us Consider the sequences $\{x_n\}$ in X such that,

$x_n = fx_{n-1} = gx_n.$

Substituting $x = x_o$ and $y = x_1$ in above inequality we obtain,

$d(x_1,x_2) = d(fx_0,gx_1)$

$\leq \alpha\, d(gx_0,gx_1) + \beta\{d(fx_0,gx_0) + d(fx_1,gx_1)\}$

$\Rightarrow d(x_1,x_2) \leq \alpha\, d(x_0,x_1) + \beta d(x_1,x_0) + \gamma\, d(x_2,x_1)$

$\Rightarrow (1-\gamma)\, d(x_1,x_2) \leq (\alpha + \beta) d(x_1,x_0)$

$\Rightarrow d(x_1,x_2) \leq \frac{\alpha+\beta}{1-\gamma} d(x_1,x_0)$

$\Rightarrow d(x_1,x_2) \leq h d(x_1,x_0)$

Where, $0 < \frac{\alpha+\beta}{1-\gamma} = h < 1$ ----------------- (2.1.1)

Again we have for $x = x_1$ and $y = x_2$

Continuing the same process we obtain,

$d(x_2,x_3) = d(fx_1,gx_2)$

$\leq \alpha\, d(gx_1,gx_2) + \beta\{d(fx_1,gx_1) + d(fx_2,gx_2)\}$

$\Rightarrow d(x_2,x_3) \leq \alpha\, d(x_1,x_2) + \beta d(x_2,x_1) + \gamma\, d(x_3,x_2)$

$\Rightarrow (1-\gamma)\, d(x_2,x_3) \leq (\alpha + \beta) d(x_1,x_2)$

$\Rightarrow d(x_2,x_3) \leq \frac{\alpha+\beta}{1-\gamma} d(x_1,x_2) = h\, d(x_1,x_2)$

$\Rightarrow d(x_2,x_3) \leq h^2\, d(x_0,x_1)$ -----------------(2.1.2)

Continuing the same process we have for the positive integer n,

$\Rightarrow d(x_n,x_{n+1}) \leq h^n\, d(x_0,x_1)$ -----------------(2.1.3)

Now applying the normality of cone we obtain from the above inequality

$\|d(x_n,x_{n+1})\| \leq K\, \|h^n\, d(x_0,x_1)\|$ ---------------(2.1.4)

Letting $n \to \infty$ we have from (2.1.4),

$\|d(x_n,x_{n+1})\| \to 0$

$\Rightarrow$ x_n is a Cauchy sequence and since X is complete cone Rectangular Metric Spaces then x_n converges to point x in X.

Now we claim that our inequality satisfies the Rectangular property.

From rectangular property we have for $x \in X$,

$d(fy,f^4y) \leq d(fy,f^2y) + d(f^2y,f^3y) + d(f^3y,f^4y)$

$\Rightarrow d(fy,f^4y) \leq h d(y,fy) + h^2\, d(fy,f^2y) + h^3 d(f^2y,f^3y)$

$\Rightarrow d(fy,f^4y) \leq \sum_{i=1}^{3} h^i\, d(y,fy)$

Similarly,

$d\left(f^2y, f^5y\right) < d\left(f^2y, f^3y\right) + d\left(f^3y, f^4y\right) + d\left(f^4y, f^5y\right)$

$\leq h^2\, d(y, fy) + h^3 d(y, fy) + h^4 d(y, fy)$

$\leq \sum_{i=2}^{4} h^i\, d(y, fy)$ -------------(2.1.5)

Thus in general for n > m and from Lemma of [4,6]

$d(f^n y, f^m y) \leq (h^{n-1} + h^{n-2} + h^{n-3} + --- h^m)\, d(y, fy)$

$\leq \frac{h^m}{1-h}\, d(y, fy)$

Now for $x_o = y \in X$, we obtain,

$d(f^n x_o, f^m x_o) \leq \frac{h^m}{1-h}\, d(x_o, fx_o)$

$\Rightarrow d(x_n, x_m) \leq \frac{h^m}{1-h}\, d(x_o, x_1)$

Applying the normality of cone we obtain,

$\|d(x_n, x_m)\| \leq \frac{h^m}{1-h} K\, \|d(x_o, x_1)\|$

Which implies that $\|d(x_n, x_m)\| \to 0$

$\Rightarrow d(x_n, x_m) \to 0$

As X is complete cone rectangular metric space then there exists a point x in (X, d) such that, $x_n \to x$ as $n \to \infty$

Now $d(x_n, fx_n) = d(fx_{n+1}, fx_n) \to 0$ as $n, m \to \infty$

Now we established that f and g has the common fixed point.

$d(x_n, gx_n) = d(fx_{n-1}, gx_n)$

$\leq \alpha\, d(gx_{n-1},gx_n) + \beta d(fx_{n-1},g x_{n-1}) + \gamma\, d(f x_n, gx_n)$

$\leq \alpha\, d(gx_{n-1},gx_n) + \beta d(x_n,gx_{n-1}) + \gamma\, d(x_{n-1}, gx_n)$

Letting $n \to \infty$ we obtain,

$d(x, gx) \leq \beta d(x,gx) + \gamma\, d(x, gx)$

$\Rightarrow (1-\beta-\gamma) d(x, gx) \leq 0$

Appling the normality of cone we obtain that,

$K\|(1-\beta-\gamma) d(x, gx)\| \leq 0$

Which is a contradiction of definition of Cone Rectangular Metric Spaces and only possibility that,

$d(x, gx) = 0$

This implies that,

$x = gx$ ------------------------(2.1.6)

That is g has a fixed point x in X.

Similarly we can prove that f has an fixed point x in X.

That is, $x = fx$. ----------------------- (2.1.7)

Now we prove that f and g has a common fixed point in X.

In the very beginning we have define for commutating mapping that,

$x_n = fx_{n-1} = gx_n$

Letting $n \to \infty$ and as X is a complete cone rectangular metric spaces we obtain,

$\lim_{n\to\infty} xn = \lim_{n\to\infty} fxn = \lim_{n\to\infty} g\, xn$

$\Rightarrow x = fx = gx$ -----------------------(2.1.8)

This shows that f and g has the common fixed point in X.

Lastly we claim that f and g has unique common fixed point.

For which we let f and g has another fixed point y in X.

i.e., $y = fy = gy$ ------------------------(2.1.9)

Therefore we have from the inequality defined in theorem,

$d(x,y) = d(fx,gy)$

$\leq \alpha\, d(gx,gy) + \beta d(fx,gx) + \gamma\, d(fy,gy)$

$\Rightarrow d(x,y) \leq \alpha\, d(x,y) + \beta d(x,x) + \gamma\, d(y,y)$ applying the equations (2.1.8),(2.1.9)

$\Rightarrow d(x,y) \leq \alpha\, d(x,y)$

$\Rightarrow (1-\alpha)\, d(x,y) \leq 0$

Applying the normality of cone metric spaces we obtain,

$\Rightarrow K\|(1-\alpha)\, d(x,y)\| \leq 0$

Which is a contradiction and only possibility that $x = y$. Hence f and g has the unique common fixed x point in X.

Theorem 2.2: Let (X, d) is a complete cone rectangular metric space and P is a normal cone with normal constant K. Let f, g and h are self mapping from X into itself and f is commutes with g and h satisfying the condition,

$d(fx,fy) \leq \alpha d(gx,gy) + \beta d(fx,gx) + \gamma d(fy,gy)$

$+ \sigma\, d(fx,hx) \quad \forall x,y \in X$ and $\alpha,\beta,\gamma \in (0,1)$also $\frac{\alpha-\beta}{1-\gamma-\delta}$ and $(1-\alpha)\in(1,0)$ Then f, g and h are unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Let us Consider the sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$y_n = fx_{n-1} = gx_n$,

and, $y_{n+1} = fx_n = hx_{n+1}$,

Now we have from the inequality defined above for $x = x_{n-1}$ and $y = x_n$

$d(y_n, y_{n+1}) = d(fx_{n-1}, fx_n)$

$\leq \alpha\, d(gx_{n-1},gx_n) + \beta\, d(f\,x_{n-1},gx_{n-1}) + \gamma\, d(fx_n,gx_n) + \sigma\, d(fx_n,hx_n)$

$\leq \alpha\, d(y_{n-1},y_n) + \beta d(y_n,y_{n-1}) + \gamma\, d(y_{n+1},y_n) + \sigma\, d(y_{n+1},y_n)$

$\Rightarrow d(y_n, y_{n+1})(1-\gamma-\sigma) \leq (\square+\beta)d(y_{n-1},y_n)$

$\Rightarrow d(y_n, y_{n+1}) \leq \frac{\alpha+\beta}{1-\gamma-\delta} d(y_{n-1},y_n)$

Taking $h = \frac{\alpha-\beta}{1-\gamma-\delta}$ we have from above inequality,

$d(y_n, y_{n+1}) \leq h d(y_{n-1},y_n)$ --------------------(2.2.1)

For m>n we have,

Now from the triangular property of cone metric spaces we have for m>n

$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+p})$

$\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+p})$

$\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+p})$

$\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) \ldots\ldots\ldots\ldots d(y_{n+p-1}, y_{n+p})$

$\leq (h^n + h^{n+1} + h^{n+2} + h^{n+3} \ldots\ldots\ldots\ldots + h^{n+p})\, d(y_0, y_1)$

$\leq h^n (1 + h + h^2 + h^3 \ldots\ldots\ldots\ldots + h^p)\, d(y_0, y_1)$

$\Rightarrow d(y_n, y_{n+p}) \leq \frac{h^n}{1-h} d(y_0, y_1)$ -------------(2.2.2)

Taking the normality of Cone, (1.5) gives,

$\| d(y_n, y_{n+p})\| \leq K \| \frac{h^n}{1-h} d(y_0, y_1) \|$

$\Rightarrow \| d(y_n, y_m)\| \leq K \| \frac{h^n}{1-h} \| \|d(y_0, y_1)\|$ --------(2.2.3)

Which yields,

$\| d(y_n, y_m)\| \to 0$ as $n, n+p \to\infty$ and $\frac{h^n}{1-h} \to 0$.

$\Rightarrow$ y_n is a Cauchy sequence and since X is complete cone Rectangular Metric Spaces then y_n converges to point y in X.

Similarly we can claim that our inequality satisfies the Rectangular property.

Now we prove that fy and gy has a common fixed point in X.

Now we have from the inequality,

$d(fx_{n-1},y_{n+1}) = d(fx_{n-1},fx_n)$

$\leq \alpha d(gx_{n-1},gx_n) + \beta d(fx_{n-1},g\,x_{n-1}) + \gamma d(fx_n,gx_n) + \sigma\, d(fx_{n-1},h\,x_{n-1})$

$\Rightarrow d(fx_{n-1},y_{n+1}) \leq \alpha\, d(gx_{n-1},gx_n) + \beta d(fx_{n-1},g\,x_{n-1}) + \gamma d(fx_n,gx_n) + \sigma\, d(fx_{n-1},h\,x_{n-1})$

$\Rightarrow d(fx_{n-1},y_{n+1}) \leq \alpha d(y_{n-1},y_n) + \beta d(fx_{n-1},y_{n-1}) + \gamma d(fx_{n-1},y_n) + \sigma\, d(fx_{n-1},y_{n-1})$

Now letting $n \to\infty$ we obtain,

$d(fy,y) \leq \alpha d(y,y) + \beta\, d(fy,y) + \gamma\, d(f\,y,y) + \sigma\, d(f\,y,y)$

This gives,

$d(fy,y)\{1-(\beta+\gamma+\delta)\} \leq 0$ --------------------(2.2.4)

Which is a contradiction of definition of cone rectangular metric spaces and as $\{1-(\beta+\gamma+\delta)\} \in (0, 1)$ and hence we have,

$d(fy,y) = 0$

$\Rightarrow fy = y$ ---------------------(2.2.5)

Similarly we can show that,

$gy = y$ --------------------(2.2.6)

Therefore we can conclude that f and g has the common fixed point y in X.

Lastly we claim that y is unique.

If possible late y is not unique. Let $y*$ is another fixed point of f and g in X.

i.e. $f\,y* = y*$ and $gy* = y*$ ----------------------(2.2.7)

Now we have from the inequality condition,

$d(fy,fy*) \leq \alpha\, d(gy,gy*) + \beta d(fy,gy) + \gamma d(fy*,gy*) + \sigma\, d(fy,hy)$

$\Rightarrow d(y,y*) \leq \alpha\, d(y,y*) + \beta d(y,y) + \gamma d(y*,y*) + \sigma\, d(y,y)$

$\Rightarrow d(y,y*)(1-\alpha) \leq 0$

Again a contradiction and hence only possibilities that,

$d(y,y*) = 0$

$\Rightarrow y = y*$ ---------------------------(2.2.8)

Hence *y* is the unique common fixed point of f, g and h in X.

CONCLUSION: In this research paper we have Realize that the self mappings f, g and the Commutating mappings f, g, h satisfies the concept of Banach fixed point condition and Cone rectangular inequality. All the self mapping and commutative mappings are an effective part to find out the existence of fixed point on complete cone Rectangular metric spaces

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